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Harmonic analysis of generalized vector functions, generalized spin-weighted functions and induced representations

P J McCarthy† and A Sudbery‡

† Department of Mathematics, King's College, The Strand, London WC2R 2LS, UK

‡ Department of Mathematics, University of York, Heslington, York, UK

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Abstract. We consider vector-valued functions on a manifold M, taking values in a representation space for a group G which also acts on M, and show how such functions can be resolved into components which transform irreducibly under G, without performing a Clebsch–Gordan decomposition. We also describe the connection with induced representations of G.

1. Introduction

It is common in physics to have to consider functions which are defined on some space subject to a group of transformations, and which take their values in a vector space which is also transformed by the same group. The most familiar example is that of vector-valued functions of position, transforming by the rotation group; the rotations act both on the points of space and on the values of the function. Gel'fand *et al* (1963) and Goldberg *et al* (1967) have pointed out that the appropriate functions to use in analysing the angular dependence of such vector-valued functions are the spin-weighted spherical harmonics, and both sets of authors have used this analysis to find a simple form for the general solution of Maxwell's equations. An alternative procedure would be to analyse each component of the vector-valued function by means of ordinary spherical harmonics, but since the components are mixed up by a rotation, a Clebsch–Gordan decomposition becomes necessary in finding sets of functions which transform irreducibly under rotations.

Gel'fand *et al* (1963) also considered the analysis of functions on the sphere with values in a general representation space of the rotation group; this is essentially the same as the helicity amplitude analysis of Jacob and Wick (1959) (indeed, 'spin-s spherical harmonics' would be better called 'helicity-s spherical harmonics').

Crampin and McCarthy (1977) have performed a similar analysis for other groups of transformations, obtaining analogues of spin-weighted spherical harmonics and showing the relation of this analysis to the theory of induced representations. The purpose of this paper is to consider the most general situation of this type and show how the most general kind of vector-valued function can be analysed into functions which transform irreducibly under the group in question, without performing a Clebsch–Gordan decomposition.

2. Analysis of general vector-valued functions

Let M be a manifold (or more generally a Hausdorff topological space) and G a group of transformations acting transitively on M. In most applications G will be a Lie group, but we need only assume that it is a locally compact topological group. The action being transitive means that every point of M can be taken to any other point by an element of G; in considering a space on which the action is not transitive, it will be necessary to decompose the space into orbits of G. In the case of ordinary three-dimensional space and rotations, this means concentrating on spheres, i.e. considering only the angular dependence of functions.

Let V be a vector space on which an irreducible representation D^0 of G is defined. We are concerned with functions $\psi\colon M\to V$ transforming under G in the way described in the introduction, with the argument transforming according to the action of G on M and the value transforming according to the representation D^0. That is, if F denotes the space of functions $\psi\colon M\to V$, we have a representation $U(g)\colon F\to F$ given by

$$[U(g)\psi](x)=D^0(g)\psi(g^{-1}x) \qquad \psi\in F,\ g\in G,\ x\in M. \tag{1}$$

We will make the further assumptions that M has a G-invariant measure $\mathrm{d}x$ which is related to left Haar measure $\mathrm{d}g$ on G by

$$\int_P f(x)\,\mathrm{d}x=\int_{P'} f(gx_0)\,\mathrm{d}g$$

where x_0 is any point of M, P is a subset of M, and $P'\subseteq M$ is the subset $\{g\colon gx_0\in P\}$; that V is a Hilbert space, and that the representation D^0 is unitary. Then we can specify F more precisely as the space of square-integrable functions, i.e. those for which

$$\int_M \langle\psi(x),\psi(x)\rangle\,\mathrm{d}x<\infty$$

where the angle brackets denote the inner product on V.

By choosing a base point $x_0\in M$, we can make functions on M correspond to functions on G as follows: given $\psi\in F$, define $\tilde\psi\colon G\to V$ by

$$\tilde\psi(g)=D^0(g^{-1})\psi(gx_0). \tag{2}$$

Then $\tilde\psi$ is a square-integrable function on G (because D^0 is unitary, and because of the relation between the measures on M and G), and it satisfies the side condition

$$\tilde\psi(gh)=D^0(h^{-1})\tilde\psi(g) \qquad \text{for all } h\in H \tag{3}$$

where H is the little group (or stability group) of x_0, i.e. the subgroup

$$H=\{g\in G\colon gx_0=x_0\}.$$

Conversely, given any such function $\tilde\psi\colon G\to V$, we can use the definition (2) to obtain a corresponding function $\psi\colon M\to V$; the side condition (3) ensures that ψ is well defined.

Let $\tilde F$ be the set of all square-integrable functions $\psi\colon G\to V$ satisfying the side condition (3). Then corresponding to the representation $U(g)$ on F we have a representation $\tilde U$ on $\tilde F$ given by

$$\tilde U(g)\tilde\psi=U(g)\psi$$

i.e.

$$[\tilde U(g)\tilde\psi](g')=D^0(g'^{-1})[U(g)\psi](g'x_0)=D^0(g'^{-1})D^0(g)\psi(g^{-1}g'x_0)=\tilde\psi(g^{-1}g').$$

Thus $\tilde{U}$ is just the regular representation on $\tilde{F}$, i.e. it acts on the functions in $\tilde{F}$ by transforming the argument by left translation, but without transforming the values of the functions. So it is appropriate to analyse each component of a function $\tilde{\psi} \in \tilde{F}$ like a scalar function. We choose a basis $\{v_i\}$ for V and write $\tilde{\psi}(g) = \Sigma \tilde{c}_i(g) v_i$; then each $\tilde{c}_i(g)$ is a square-integrable scalar function on G and we can analyse it as

$$\tilde{c}_i(g) = \sum_{\sigma \in \hat{G}} \mathrm{Tr}[A_i^\sigma D^\sigma(g^{-1})] = \sum_{\sigma, mn} A^\sigma_{mn,i} D^\sigma_{mn}(g)$$

where $\hat{G}$ denotes the set of unitary irreducible representations of G, $D^\sigma(g)$ is the matrix representing g in the representation σ, and A_i^σ is a matrix of coefficients, of the same size as $D^\sigma(g)$.

This expansion of $\tilde{c}_i(g)$ gives us an expansion of $\tilde{\psi}$ in terms of the functions

$$F^\sigma_{mn,i}(g) = \overline{D^\sigma_{mn,i}(g)} v_i.$$

However, these functions do not satisfy the side condition, i.e. they do not correspond to functions on M. We can obtain functions which do satisfy the side condition by integrating over H, defining

$$\tilde{\psi}^\sigma_{mn,i}(g) = \int_H \overline{D^\sigma_{mn}(gh)} D^0(h) v_i \, \mathrm{d}h$$

where $\mathrm{d}h$ denotes left Haar measure on H (so that $\mathrm{d}(h_0 h) = \mathrm{d}h$), normalized so that $\int_H \mathrm{d}h = 1$. Then these functions correspond to functions on M, which can be written explicitly as

$$\psi^\sigma_{mn,i}(x) = \int_H \overline{D^\sigma_{mn}(\gamma_x h)} D^0(\gamma_x h) v_i \, \mathrm{d}h$$

where γ_x is a 'Wigner boost', i.e. an arbitrarily chosen group element which takes x_0 to x.

To show that any function $\psi \in F$ can be expanded in terms of the $\psi^\sigma_{mn,i}$, note that $\psi(ghx_0)$ is independent of $h \in H$ and so

$$\psi(x) = \psi(\gamma_x x_0) = \int_H \psi(\gamma_x h x_0) \, \mathrm{d}h = \int_H D^0(\gamma_x h) \tilde{\psi}(\gamma_x h) \, \mathrm{d}h$$

$$= \int_H D^0(\gamma_x h) \sum_{\sigma, mni} A^\sigma_{mn,i} \overline{D^\sigma_{mn}(\gamma_x h)} v_i \, \mathrm{d}h = \sum_{\sigma, mni} A^\sigma_{mn,i} \psi^\sigma_{mn,i}(x).$$

It follows from the construction of the $\psi^\sigma_{mn,i}$, and can easily be checked directly, that if σ, n and i are fixed the functions $\psi^\sigma_{mn,i}$ transform under U according to the irreducible representation σ of G.

The functions $\tilde{\psi}^\sigma_{mn,i}$ are not independent, since they are obtained by projecting a basis for the larger space of all square-integrable functions on G onto the subspace $\tilde{F}$. The effect of the integration over H is to pick out the parts of the matrix elements $D^\sigma_{mn}(g)$ which transform under H according to one of the irreducible representations contained in the restriction of D^0 to H; this enables us to find a linearly independent subset of the functions $\tilde{\psi}^\sigma_{mn,i}$.

From the definition of $\tilde{\psi}^{\sigma}_{mn,i}$ we have

$$\tilde{\psi}^{\sigma}_{mn,i}(g)=\sum_p \overline{D^{\sigma}_{mp}(g)}\phi^{\sigma}_{pn,i}$$

where

$$\phi^{\sigma}_{mn,i}=\int_H D^{\sigma}_{mn}(h)D^0(h)v_i\,\mathrm{d}h.$$

Let $\{u_m\}$ be the basis of the representation space V^{σ} of D^{σ} with respect to which the matrix elements are defined, so that $D^{\sigma}_{mn}(g)=\langle u_m|D^{\sigma}(g)|u_n\rangle$, and suppose this basis is adapted to H in the sense that each u_m belongs to a subspace $V^{\sigma}_{\alpha a}$ on which D^{σ} acts like an irreducible representation T^{α} of H (this is possible since D^{σ} is unitary and therefore completely reducible when restricted to H). The label a is included to account for possible multiplicity of the representation T^{α}. Then $D^{\sigma}_{mn}(h)=0$ unless u_m belongs to the same subspace $V^{\sigma}_{\alpha a}$ as u_n, and then $D^{\sigma}_{mn}(h)=T^{\alpha}_{mn}(h)$. We reduce the representation D^0 similarly, decomposing the space V into subspaces $V^0_{\alpha a}$ on which D^0 acts like the irreducible representation T^{α}. Then we have

$$D^0(h)v_i=\sum_{j\in\alpha a} T^{\alpha}_{ji}(h)v_j \qquad \text{if } i\in\alpha a$$

where we write $i\in\alpha a$ as an abbreviation for $v_i\in V^0_{\alpha a}$ (and similarly we will write $m\in\alpha a$ for $v_m\in V^{\sigma}_{\alpha a}$).

It follows that $\phi^{\sigma}_{mn,i}=0$ unless u_m and u_n belong to the same subspace $V^{\sigma}_{\alpha a}$, and then

$$\phi^{\sigma}_{mn,i}=\sum_{j\in\beta b}\int_H \overline{T^{\alpha}_{mn}(h)}T^{\beta}_{ji}(h)\,\mathrm{d}h\,v_j$$

where $V^0_{\beta b}$ is the H-invariant subspace of V containing v_i. By the orthogonality of the matrix elements of the irreducible representations of H, this becomes

$$\phi^{\sigma}_{mn,i}=\sum_{j\in\beta b}\delta_{\alpha\beta}\delta_{mj}\delta_{ni}v_j=\delta_{\alpha\beta}\delta_{ni}v_m.$$

Now consider

$$\tilde{\psi}^{\sigma}_{mn,i}(g)=\sum_p \overline{D^{\sigma}_{mp}(g)}\phi^{\sigma}_{pn,i}.$$

Suppose $u_n\in V^{\sigma}_{\alpha a}$ and $v_i\in V^0_{\beta b}$. Then $\phi^{\sigma}_{pn,i}=0$ unless $\alpha=\beta$ and $p\in\alpha a$; thus $\tilde{\psi}^{\sigma}_{mn,i}(g)=0$ unless $\alpha=\beta$, and then

$$\tilde{\psi}^{\sigma}_{mn,i}(g)=\sum_p \overline{D^{\sigma}_{mp}(g)}\delta_{ni}v_p=\sum_p \overline{\langle u_m|D^{\sigma}(g)|u_p\rangle}\delta_{ni}v_p$$

where the sum is taken over all basis vectors $u_p\in V^{\sigma}_{\alpha a}$ and $v_p=V^0_{\beta b}$, u_p and v_p being corresponding vectors in the isomorphic subspaces $V^{\sigma}_{\alpha a}$ and $V^0_{\beta b}$. Thus the only non-zero $\tilde{\psi}^{\sigma}_{mn,i}(g)$ are those for which the H-invariant subspace $V^{\sigma}_{\alpha a}\subseteq V^{\sigma}$ containing u_n and the H-invariant subspace $V^0_{\alpha b}\subseteq V$ containing v_i transform by equivalent irreducible representations of H, and for which u_n and v_i correspond to each other under the isomorphism between $V^{\sigma}_{\alpha a}$ and $V^0_{\alpha b}$ which defines the equivalence. Moreover, when this requirement is satisfied the functions are equal for all values of $i \in \alpha b$; so we need only consider the functions

$$\tilde{Y}^{\sigma}_{m\alpha ab}(g) = \sum_n D^{\sigma}_{mn}(g) v_n$$

the sum being taken over all $u_n \in V^{\sigma}_{\alpha a}$ and $v_n \in V^0_{\alpha b}$. Because the $D^{\sigma}_{mn}(g)$ are independent functions on G and the v_n are independent vectors in V, it can be seen (look hard!) that the $\tilde{Y}^{\sigma}_{m\alpha ab}(g)$ are independent functions in $\tilde{F}$. They correspond to functions $\tilde{Y}^{\sigma}_{m\alpha ab}$: $M \to V$ given by

$$Y^{\sigma}_{m\alpha ab}(x) = \sum_n \overline{D^{\sigma}_{mn}(\gamma_x)} D^0(\gamma_x) v_n.$$

These are our generalized spin-weighted spherical harmonics. They are a basis for the space of square-integrable functions ψ: $M \to V$, and for fixed α, a and b the functions $Y^{\sigma}_{m\alpha ab}$ transform according to the irreducible representation D^{σ} of G. We call these functions *D^0-adapted harmonics.*

3. Generalized spin-weighted functions and induced representations

Ordinary spin-s spherical harmonics are examples of spin-weighted functions on the sphere S^2—in fact they form a basis for the space of all such functions—and Crampin and McCarthy (1977) have shown that spin-weighted functions on S^2 are just the functions occurring (as the representation space) in the representation of SO(3) induced from a representation of the subgroup SO(2). They have also generalized the notion of spin-weighted function so as to apply to a homogeneous space of any group G (i.e. a space on which G acts transitively) in such a way as to maintain the connection with induced representations. In this section we will review the notion of an induced representation from a slightly different standpoint from that of Crampin and McCarthy, in order to find the relation between the spin-weighted spherical harmonics introduced in § 2 and their generalized spin-weighted functions. We will see that our spin-weighted spherical harmonics form a basis for a special kind of spin-weighted function, and a simple modification provides a basis for the most general spin-weighted functions.

Induced representations of groups are exemplified by Wigner's construction of the irreducible representation of the Poincaré group; the representation of the Poincaré group describing a particle of (real) mass m and spin s is the representation of the Lorentz group (which plays the role of G) induced from the spin-s representation of the rotation group (which plays the role of H). The role of the manifold M is played by the mass shell $p^2 = m^2$. In the general case, we consider a vector space W_x associated with each $x \in M$ and suppose that a group element that takes x to y also takes W_x to W_y; thus for each $x \in M$ and $g \in G$ we have an isomorphism $\Gamma_x(g)$: $W_x \to W_{gx}$. We further suppose that

$$\Gamma_{g_2 x}(g_1)\Gamma_x(g_2) = \Gamma_x(g_1 g_2).$$

Then the mappings $\Gamma_x(h)$, where h belongs to the little group H_x of x, form a representation of H_x on W_x. Now all the little groups of the points of M are isomorphic, since $H_{gx} = gH_x g^{-1}$, and all the representations of the little groups are equivalent, since

$$\Gamma_{gx}(ghg^{-1})\Gamma_x(g) = \Gamma_x(g)\Gamma_x(h)$$

so that $\Gamma_x(g)$ intertwines the representations of the little groups of x and gx. Let H be a standard model of the little groups H_x, so that for each x we have an isomorphism $\kappa_x: H_x \to H$, and let T^0 be the representation of H which is equivalent to the representations Γ_x. Then if T^0 acts on a vector space W, we have isomorphisms $\lambda_x: W_x \to W$ such that

$$\lambda_x^{-1} T^0(h) \lambda_x = \Gamma_x(\kappa_x(h))$$

(for example, we could take $H = H_{x_0}$, $\kappa_x(h) = \gamma_x h \gamma_x^{-1}$, $T^0 = \Gamma_{x_0}$ and $\lambda_x = \Gamma_{x_0}(\gamma_x)$, where x_0 is a base point in M and γ_x is a Wigner boost). We define a T^0-*weighted function* on M to be a function $\hat{\psi}: M \to \bigcup_{x\in M} W_x$ such that $\psi(x) \in W_x$. In other words, a T^0-weighted function is a cross section of the vector bundle $B = \bigcup_{x\in M} W_x$.

To make contact with the original definition of spin-weighted functions (Newman and Penrose 1966), choose a basis $v_1(x), \ldots, v_n(x)$ for each W_x. Then a cross section $\hat{\psi}$ is specified by giving the coordinates $c_1(x), \ldots, c_n(x)$ of $\hat{\psi}(x)$ with respect to this basis. If we change basis in W_x by means of an element of the little group H_x, so that the new basis vectors are

$$v_i'(x) = \Gamma_x(h_x) v_i(x) \qquad \text{for some } h_x \in H_x$$

then the coordinates with respect to the new basis will be

$$c_i'(x) = \sum_j T_{ij}^0(k_x) c_j(x) \tag{4}$$

where $k_x = \kappa_x(h_x) \in H$ and $T_{ij}^0(k_x)$ are the matrix elements of $T^0(k_x)$ with respect to the basis $\lambda_x v_i(x)$ of W. Thus a T^0-weighted function on M could be defined as a set of n scalar functions which transform under elements of the little groups according to (4). In the case of spin-weighted functions on S^2, the little group H_x is the group of rotations of the tangent plane at x, the space W_x is a one-dimensional complex space, and the representation T^0 is that which associates a rotation through an angle α with multiplication by $e^{is\alpha}$; hence the definition of a spin-weighted function as a complex-valued function $c(x)$ which transforms under rotation of the tangent plane at x through $\alpha(x)$ by

$$c'(x) = e^{is\alpha(x)} c(x).$$

In what follows we will assume that we have chosen a base point $x_0 \in M$, and we will identify H with the subgroup H_{x_0} and W with the fibre W_{x_0}.

The representation of G *induced* by the representation T of H acts on the space of square-integrable T^0-weighted functions on M, and is defined by

$$[\hat{U}(g)\hat{\psi}](x) = \Gamma_x(g)\hat{\psi}(g^{-1}x).$$

If we identify all the fibres W_x with W_{x_0} by means of Wigner boosts γ_x, we can replace cross sections $\hat{\psi}$ by functions $\psi: M \to W$, the correspondence being given by

$$\psi(x) = \Gamma_x(\gamma_x^{-1})\hat{\psi}(x).$$

To the induced representation $\hat{U}(g)$ there corresponds a representation $U(g)$ defined on the space F_1 of square-integrable functions $\psi: M \to V$; it acts by

$$[U(g)\psi](x) = [\hat{U}(g)\hat{\psi}](x).$$

Thus

$$[U(g)\psi](x)=\Gamma_x(\gamma_x^{-1})[\hat{U}(g)\hat{\psi}](x)=\Gamma_x(\gamma_x^{-1})\Gamma_{g^{-1}x}(g)\hat{\psi}(g^{-1}x)$$
$$=\Gamma_x(\gamma_x^{-1})\Gamma_{g^{-1}x}(g)\Gamma_{x_0}(\gamma_{g^{-1}x})\psi(g^{-1}x)=T^0[h(g,x)]\psi(g^{-1}x) \quad (5)$$

where $h(g,x)=\gamma_x^{-1}g\gamma_{g^{-1}x}$ is a 'Wigner rotation'.

We now make functions $\psi\in F_1$ correspond to functions $\tilde{\psi}: G\to W$ by defining

$$\tilde{\psi}(g)=T^0(g^{-1}\gamma_{gx_0})\psi(gx_0). \quad (6)$$

Then $\tilde{\psi}$ is a square-integrable function on G and satisfies the side condition

$$\tilde{\psi}(gh)=T^0(h^{-1})\tilde{\psi}(g). \quad (7)$$

Conversely, given any such function $\tilde{\psi}: G\to W$, we can use the definition (6) to obtain a corresponding function $\psi: M\to W$; the side condition (7) ensures that ψ is well defined.

Proceeding as in § 2, we find that the representation U corresponds to a representation $\tilde{U}$ given by

$$[U(g)\tilde{\psi}](g')=\tilde{\psi}(g^{-1}g').$$

Thus the situation is exactly the same as in § 2, except that the functions $\tilde{\psi}$ take their values in a vector space W which carries a representation of the subgroup H only, instead of the whole group G. However, in the analysis of the functions ψ in § 2 we only needed to consider the action of the representation D^0 for elements of the subgroup H (in fact the representation $U(g)$ considered in § 2 is equivalent to the representation induced from the restriction of D^0 to H). Hence that analysis can be taken over to the present situation, yielding the conclusion that a basis for square-integrable functions $\tilde{\psi}: G\to W$ is given by the functions

$$\tilde{Z}^{\sigma}_{m\alpha ab}(g)=\sum_n \overline{D^{\sigma}_{mn}(g)w_n}$$

where the sum is taken over all basis vectors $w_n\in W$ which lie in an invariant subspace W_{ab} on which the representation T^0 acts like the irreducible representation T^{α}, and the matrix elements $D^{\sigma}_{mn}(g)$ refer to the basis vectors corresponding to w_n in an H-invariant subspace $V^{\sigma}_{\alpha a}$ of the representation space of D^{σ}.

Passing to functions on M by means of the correspondence (4), we obtain basis functions

$$Z^{\sigma}_{m\alpha ab}(x)=\sum_n \overline{D^{\sigma}_{mn}(\gamma_x)}w_n.$$

If the representation T^0 is irreducible of the type T^{α}, the label b becomes unnecessary and we obtain *irreducible α-weighted harmonics* $Z^{\sigma}_{m\alpha a}(x)$. These form a basis for the space of square-integrable α-weighted functions on M, and for fixed α, a and σ they transform according to the irreducible representation σ of G under the representation (4) induced from T^0.

The relation between the irreducible α-weighted harmonics $Z^{\sigma}_{m\alpha a}(x)$ and the D^0-adapted harmonics $Y^{\sigma}_{m\alpha ab}$ of § 2 is

$$Y^{\sigma}_{m\alpha ab}=\iota_b D^0(\gamma_x)Z^{\sigma}_{m\alpha a}(x)$$

where $\iota_b: W^{\alpha}\to V^0_{\alpha b}$ injects the representation space W^{α} on which $T^{\alpha}(h)$ acts into the space V^0 on which $D^0(g)$ acts. The factor $D^0(\gamma_x)$ is present because the representation induced from the restriction of D^0 to H is not identical to the representation (1), but is equivalent to it; the two representations are intertwined by the mapping $\psi(x) \to D^0(\gamma_x)\psi(x)$. This relation between the two representations is the same as that between Dirac's and Wigner's description of spin-$\frac{1}{2}$ particles, as explained by Joos (1962).

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